THE CHINESE UNIVERSITY OF HONG KONG
DEPARTMENT OF MATHEMATICS

MATH1010 I/J University Mathematics 2015-2016
Suggested Solution to Problem Set 4

1. (a)

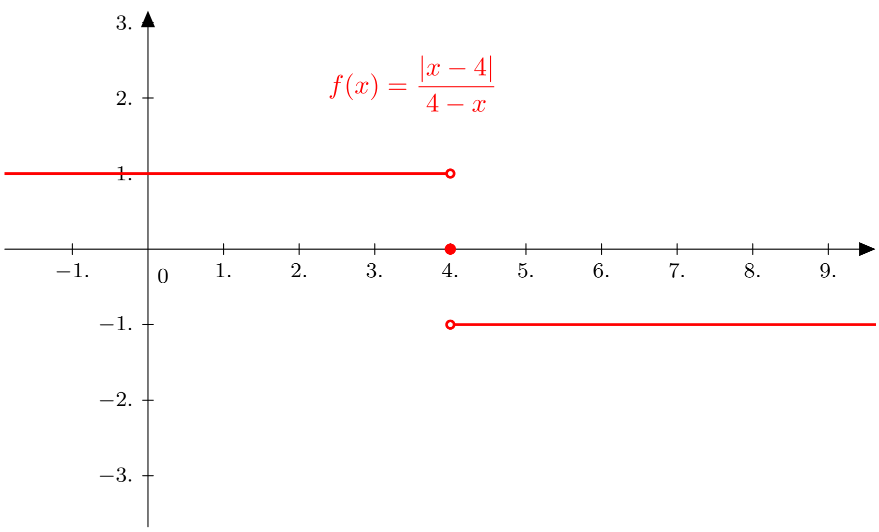
$$\begin{aligned}
\lim_{x\to+\infty}\left(\frac{x+1}{x-1}\right)^x &= \lim_{x\to+\infty}\left(1+\frac{2}{x-1}\right)^x \\
&= \lim_{x\to+\infty}\left(\left(1+\frac{1}{(\frac{x-1}{2})}\right)^{\frac{x-1}{2}}\right)^2\left(1+\frac{1}{(\frac{x-1}{2})}\right)^{\frac{1}{2}} \\
&= (e^2)(1) \\
&= e^2
\end{aligned}$$

(b)

$$\begin{aligned}
\lim_{x\to+\infty}\left(\frac{x^2-2x-3}{x^2-3x-28}\right)^x &= \lim_{x\to+\infty}\left(1+\frac{x+25}{x^2-3x-28}\right)^x \\
&= \lim_{x\to+\infty}\left(1+\frac{1}{(\frac{x^2-3x-28}{x+25})}\right)^{\frac{x^2-3x-28}{x+25}}\left(1+\frac{1}{(\frac{x^2-3x-28}{x+25})}\right)^{\frac{28x+28}{x+25}} \\
&= (e)(1) \\
&= e
\end{aligned}$$

2. (a) Note that $f(x)$ can be reformulated as the following:

$$f(x)=\begin{cases} 1 & \text{if} \quad x<4; \\ 0 & \text{if} \quad x=4; \\ -1 & \text{if} \quad x>4. \end{cases}$$



(b) Note that $\lim_{x\to 4^+} f(x) = \lim_{x\to 4^+} -1 = -1$ and $\lim_{x\to 4^-} f(x) = \lim_{x\to 4^-} 1 = 1$.
Therefore, $\lim_{x\to 4} f(x)$ does not exist and $f(x)$ is not continuous at $x = 4$.

3. For $x \neq 0$, we have

$$\begin{array}{rcl} -1 \leq & \cos(\frac{1}{e^x-1}) & \leq 1 \\ -x^2 \leq & x^2\cos(\frac{1}{e^x-1}) & \leq x^2 \\ -x^2 \leq & f(x) & \leq x^2 \end{array}$$

Since $\lim_{x\to 0} -x^2 = \lim_{x\to 0} x^2 = 0$, by the sandwich theorem $\lim_{x\to 0} f(x) = 0$.
We have $\lim_{x\to 0} f(x) = f(0)$, so $f(x)$ is continuous at $x = 0$.

4. (a) Put $x = y = 0$, we have $f(0) = [f(0)]^2$ which implies $f(0) = 0$ or 1. Since $f(0) \neq 0$, $f(0) = 1$.

(b) Since $f(x)$ is continuous at $x = 0$, we have

$$\begin{aligned} \lim_{h\to 0} f(0+h) &= f(0) \\ \lim_{h\to 0} f(h) &= 1 \end{aligned}$$

Now, let $x_0 \in \mathbb{R}$.

$$\begin{aligned} \lim_{h\to 0} f(x_0+h) &= \lim_{h\to 0} f(x_0)f(h) \\ &= f(x_0)\left(\lim_{h\to 0} f(h)\right) \\ &= f(x_0)\cdot 1 \\ &= f(x_0) \end{aligned}$$

Therefore, $f(x)$ is continuous at $x = x_0$.

Since x_0 is an arbitrary point, it means $f(x)$ is continuous everywhere.

5. (a) Put $x = y = 1$, we have $f(1) = 2f(1)$ and so $f(1) = 0$.

(b) Let m be a natural number. $f(a^m) = f(a\cdot a\cdots a) = f(a)\cdot f(a)\cdots f(a) = [f(a)]^m$.
Let r be a positive rational number, then $r = \dfrac{m}{n}$ where m and n are natural numbers.
By the previous result,

$$\begin{aligned} f(a^m) &= f((a^{m/n})^n) \\ &= nf(a^{m/n}) \\ \frac{1}{n}f(a^m) &= f(a^{m/n}) \\ \frac{m}{n}f(a) &= f(a^{m/n}) \\ rf(a) &= f(a^r) \end{aligned}$$

Let q be a negative rational number. We have

$$f(a^q) + f(a^{-q}) = f(a^q \cdot a^{-q}) = f(1) = 0.$$

Note that $-q$ is a positive rational number, therefore,

$$f(a^q) = -f(a^{-q}) = -(-q)f(a) = qf(a).$$

Combining the above cases and that $f(a^0) = f(1) = 0 = 0f(a)$, the result follows.

(c) Let $x \in \mathbb{R}$ and $\{x_n\}$ be a sequence with $\lim_{n\to\infty} x_n = x$.

Since f and power function are continuous, we have

$$\lim_{n\to\infty} f(a^{x_n}) = f(\lim_{n\to\infty} a^{x_n}) = f(a^{\left(\lim_{n\to\infty} x_n\right)}) = f(a^x).$$

Also, from (b),

$$\lim_{n\to\infty} f(a^{x_n}) = \lim_{n\to\infty} x_n f(a) = xf(a).$$

Therefore, for any $x \in \mathbb{R}$ and $a > 0$, $f(a^x) = xf(a)$.

Next, let $y = a^x$, for $x > 0$. Then $x = \dfrac{\ln y}{\ln a}$ and so

$$f(y) = \frac{\ln y}{\ln a} f(a) = \frac{f(a)}{\ln a} \ln y.$$

By replacing y and $\dfrac{f(a)}{\ln a}$ by x and c respectively, then $f(x) = c \ln x$.

(Remark: Therefore, all continuous functions that satisfy the condition $f(xy) = f(x) + f(y)$ for all $x, y > 0$ must be in the form $f(x) = c \ln x$ for some constant c.)